

# Dynamics of interacting qubits coupled to a common bath: Non-Markovian quantum state diffusion approach

Xinyu Zhao\*, Jun Jing, Brittany Corn and Ting Yu†

*Center for Controlled Quantum Systems and the Department of Physics and Engineering Physics,  
Stevens Institute of Technology, Hoboken, New Jersey 07030, USA*

Non-Markovian dynamics is studied for two interacting qubits strongly coupled to a dissipative bosonic environment. For the first time, we have derived the non-Markovian quantum state diffusion (QSD) equation for the coupled two-qubit system without any approximations, and in particular, without the Markov approximation. As an application and illustration of our derived time-local QSD equation, we investigate the temporal behavior of quantum coherence dynamics. In particular, we find a strongly non-Markovian regime where entanglement generation is significantly modulated by the environmental memory. Additionally, we studied the residual entanglement in the steady state by analyzing the steady state solution of the QSD equation. Finally, we have discussed an approximate QSD equation.

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## I. INTRODUCTION

The theory of quantum open systems has attracted growing attention because the framework of system plus environment has become a fundamental paradigm in quantum dissipative system, quantum optics and quantum information science [1, 2]. In reality, no quantum systems can be completely isolated from their environment. The mutual interaction between the system and environment is responsible for many important physical processes such as dissipation, fluctuation, as well as decoherence and disentanglement. Quantum open system is also an essential part of quantum information processing where the quantum noise and control are major issues in realizing quantum computing, quantum teleportation and quantum cryptography [3].

While the theory of open systems in the Markov regimes can be extensively treated with the standard Lindblad master equations [1] or the corresponding Markov quantum trajectories [4–7], the microscopic non-Markovian theory has not been well developed for the systems consisting of interacting spins, which are of experimental interest [8]. The difficulty arising from the strong coupling between the system and its environment is that the non-Markovian effects due to action and back reaction between the system and the environment must be taken into account. Undoubtedly, some memory related properties such as quantum entanglement, inhibited spontaneous emission, quantum transport, quantum tunneling [9–17] which are of importance will be lost when simply taking the Markov limit. Among all of the techniques attempt to investigate the non-Markovian evolution, the quantum state diffusion (QSD) equation initially proposed by Diósi, Gisin, and Strunz [18] pro-

vides a powerful tool to deal with the quantum open systems coupled to a boson bath [19–21].

The recent developments of the non-Markovian dynamics are concerned with a theoretical derivation of the QSD equation directly from an underlying microscopic model irrespective of environmental memory time and coupling strength [18]. The non-Markovian QSD equation is equivalent to the master equation formalism, but offers numerical advantages over the corresponding master equation if it exists. Typical to non-Markovian evolution is that the dynamics is explicitly dependent on the past history manifested in the integrals over the time in the dynamic equation [22–24]. In the case of non-Markovian QSD, under certain conditions, the nonlocal integral can be cast into a time-local form [18, 21, 25, 26]. Then, the time-local QSD equation can be efficiently implemented as an analytical and numerical tools to describe the non-Markovian effects.

The primary purpose of this paper is to study the non-Markovian quantum dynamics of interacting qubits. We derive a closed exact time-local QSD equation for two interacting qubits coupled to a bosonic environment by replacing the functional derivative in the QSD equation with a time-dependent operator called  $O$  operator. We have established the explicit equations of motion for the coefficient functions contained in the  $O$  operator. The second purpose of this paper is to investigate the non-Markovian dynamics of qubit systems by using the derived exact QSD equation. Since the exact QSD equation is valid for an arbitrary structured boson bath and the strong coupling regime, we are thus able to address the important issue of entanglement generation via environment memory in the full non-Markovian regime. It is shown that several interesting features may arise from this coupled qubit system when we examine the dynamics in a non-Markovian regime. With this coupled qubit model, we have studied the approximation based on the functional expansion of the  $O$  operator, and reveal some interesting features on non-Markovian perturbation theory.

\*Email: xzhao1@stevens.edu

†Email: Ting.Yu@stevens.edu

The organization of this paper is as follows. In Sec. II, we introduce an interacting qubit model and develop the theory of the exact time-local quantum state diffusion equation. In Sec. III, based on the exact QSD equation derived, we consider memory-assisted entanglement to illustrate the implications of non-Markovian dynamics on entanglement generation. As an example, we show that entanglement generation may be significantly modified by the environment memory time. With the quantum trajectories generated by the exact QSD equation, we also analyze the steady state solution for the single trajectories recovering the steady state entanglement described by the density matrix. We conclude in Sec. IV.

## II. THE MODEL AND TIME-LOCAL QUANTUM STATE DIFFUSION EQUATION

The model consists of a pair of coupled two-level atoms (or spins) interacting with a common bath. The total Hamiltonian can be written as (setting  $\hbar = 1$ ),

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{env}} + H_{\text{int}}, \quad (1)$$

where

$$\begin{aligned} H_{\text{sys}} &= \omega_A \sigma_z^A + \omega_B \sigma_z^B + J_{xy}(\sigma_+^A \sigma_-^B + \sigma_-^A \sigma_+^B) + J_z \sigma_z^A \sigma_z^B, \\ H_{\text{env}} &= \sum_j \omega_j b_j^\dagger b_j, \\ H_{\text{int}} &= \sum_j (g_j b_j^\dagger L + g_j^* b_j L^\dagger), \end{aligned} \quad (2)$$

where  $L = \kappa_A \sigma_-^A + \kappa_B \sigma_-^B$  is the system Lindblad operator coupled to the environment. Note that  $\kappa_A, \kappa_B$  are constants describing different coupling strengths for two spins.

This type of interaction between qubits has been studied widely in the Heisenberg spin chain models [27–29] and proved to be a useful model in quantum information processing based on spin chain. Moreover, some cavity-QED systems may be described by a similar type of Hamiltonian in simulating the Heisenberg spin chain [30]. Notably, our model in the framework of quantum open system is a very general case, including many other physically interesting models as special cases of this Hamiltonian. For example, when  $J_z = 0$ , this model will reduce to the example of the Heisenberg  $XX$  type interaction. When  $J_{xy} = J_z = 0$ , two qubits will be treated as non-interacting spin. When  $\omega_B = J_{xy} = J_z = \kappa_B = 0$ , this model simply reduces to the one qubit dissipative model [19].

The stochastic Schrödinger equation for this interacting spin model may be formally written as [18, 19, 21],

$$\frac{\partial}{\partial t} \psi_t = -i H_{\text{sys}} \psi_t + L z_t^* \psi_t - L^\dagger \int_0^t ds \alpha(t, s) \frac{\delta \psi_t}{\delta z_s^*}, \quad (3)$$

where  $\alpha(t, s) = \sum_j |g_j|^2 e^{-i\omega_j(t-s)}$  is the bath correlation function and  $z_t^* = -i \sum_j g_j^* z_j^* e^{i\omega_j t}$  is a complex

Gaussian process satisfying  $M[z_t] = M[z_t z_s] = 0$ , and  $M[z_t^* z_s] = \alpha(t, s)$ . Here  $M[\cdot]$  denotes the ensemble average over the classical noise  $z_t$ . The quantum trajectory  $\psi_t(z^*)$  recovers the density operator of the system in the ensemble average:  $\rho_t = M[|\psi_t(z^*)\rangle\langle\psi_t(z^*)|] = \int \frac{dz^2}{\pi} e^{-|z|^2} |\psi_t(z^*)\rangle\langle\psi_t(z^*)|$ .

The above exact equation contains a time-nonlocal term, it renders practical application impossible without evoking certain approximation. The existence of the time-local equation depends on if one can replace the functional derivative in the integral in Eq. (3) with a time-dependent (even noise-dependent) operator  $O(t, s, z^*)$ :

$$\frac{\delta \psi_t(z^*)}{\delta z_s^*} = O(t, s, z^*) \psi_t(z^*). \quad (4)$$

Using the “consistency condition” [18]

$$\frac{\delta}{\delta z_s^*} \frac{\partial \psi_t}{\partial t} = \frac{\partial}{\partial t} \frac{\delta \psi_t}{\delta z_s^*}, \quad (5)$$

one can obtain a formal evolution equation for the operator  $O(t, s, z^*)$ :

$$\frac{\partial}{\partial t} O = [-i H_{\text{sys}} + L z_t^* - L^\dagger \bar{O}, O] - L^\dagger \frac{\delta}{\delta z_s^*} \bar{O}, \quad (6)$$

where  $\bar{O}(t, z^*) \equiv \int_0^t ds \alpha(t-s) O(t, s, z^*)$ . This equation of motion for the  $\bar{O}$  operator has to be solved with the initial condition,

$$O(t, s = t, z^*) = L. \quad (7)$$

The explicit  $O$  operator in Eq. (6) has been determined for several interesting models [18, 21, 25]. However, the construction of the  $O$  operator for the coupled two spins has not been found before. It is generally very difficult to find the exact  $O$  operator. As the major result of this paper, we have found the exact  $O$  operator for this coupled qubit model. To begin with, we note that the  $O$  operator may be expanded in the following way,

$$\begin{aligned} O(t, s, z^*) &= f_1(t, s) O_1 + f_2(t, s) O_2 + f_3(t, s) O_3 \\ &+ f_4(t, s) O_4 + i \int_0^t ds' f_5(t, s, s') z_{s'}^* O_5, \end{aligned} \quad (8)$$

where the basis operators are given by

$$\begin{aligned} O_1 &= \sigma_-^A, \quad O_2 = \sigma_-^B, \quad O_3 = \sigma_z^A \sigma_-^B, \\ O_4 &= \sigma_z^B \sigma_-^A, \quad O_5 = 2\sigma_-^A \sigma_-^B, \end{aligned} \quad (9)$$

and  $f_j$  ( $j = 1, 2, 3, 4, 5$ ) are some time-dependent coefficients. Substituting Eq. (8) into Eq. (6), we obtain the partial differential equations governing the coefficients of the  $O$  operator.

$$\begin{aligned}
\frac{\partial}{\partial t} f_1(t, s) &= +2i\omega_A f_1 + \kappa_A F_1 f_1 + \kappa_B F_3 f_1 - iJ_{xy} f_3 - \kappa_B F_1 f_3 + \kappa_B F_4 f_3 + 2iJ_z f_4 + \kappa_A F_4 f_4 + \kappa_B F_3 f_4 - i\kappa_B F_5, \\
\frac{\partial}{\partial t} f_2(t, s) &= +2i\omega_B f_2 + \kappa_A F_4 f_2 + \kappa_B F_2 f_2 + 2iJ_z f_3 + \kappa_A F_4 f_3 + \kappa_B F_3 f_3 - iJ_{xy} f_4 - \kappa_A F_2 f_4 + \kappa_A F_3 f_4 - i\kappa_A F_5, \\
\frac{\partial}{\partial t} f_3(t, s) &= -iJ_{xy} f_1 - \kappa_A F_2 f_1 + \kappa_A F_3 f_1 + 2iJ_z f_2 + \kappa_A F_4 f_2 + \kappa_B F_3 f_2 + 2i\omega_B f_3 + \kappa_A F_4 f_3 + \kappa_B F_2 f_3 - i\kappa_A F_5, \\
\frac{\partial}{\partial t} f_4(t, s) &= +2iJ_z f_1 + \kappa_A F_4 f_1 + \kappa_B F_3 f_1 - iJ_{xy} f_2 - \kappa_B F_1 f_2 + \kappa_B F_4 f_2 + 2i\omega_A f_4 + \kappa_A F_1 f_4 + \kappa_B F_3 f_4 - i\kappa_B F_5, \\
\frac{\partial}{\partial t} f_5(t, s, s') &= +\kappa_A F_5 f_1 + \kappa_B F_5 f_2 - \kappa_B F_5 f_3 - \kappa_A F_5 f_4 + 2i\omega_A f_5 + 2i\omega_B f_5 + \kappa_A F_1 f_5 + \kappa_A F_4 f_5 + \kappa_B F_2 f_5 + \kappa_B F_3 f_5,
\end{aligned} \tag{10}$$

where  $F_j(t) = \int_0^t ds \alpha(t, s) f_j(t, s)$  ( $j = 1, 2, 3, 4$ ) and  $F_5(t, s') = \int_0^t ds \alpha(t, s) f_5(t, s, s')$ , with the initial conditions for Eq. (10):

$$\begin{aligned}
f_1(t, s = t) &= \kappa_A, \\
f_2(t, s = t) &= \kappa_B, \\
f_3(t, s = t) &= 0, \\
f_4(t, s = t) &= 0, \\
f_5(t, s = t, s') &= 0, \\
f_5(t, s, s' = t) &= -i[\kappa_A f_3(t, s) + \kappa_B f_4(t, s)].
\end{aligned} \tag{11}$$

The time evolution for this model can be solved numerically with the coefficients determined by Eq. (10).

With the explicit  $O$  operator (8), the linear QSD equation can be written compactly as a time-local equation:

$$\frac{\partial}{\partial t} \psi_t = -iH_{\text{sys}} \psi_t + L z_t^* \psi_t - L^\dagger \bar{O}(t, z^*) \psi_t, \tag{12}$$

where

$$\bar{O}(t, z^*) = \sum_{j=1}^4 F_j(t) O_j + i \int_0^t ds' F_5(t, s') z_s^* O_5. \tag{13}$$

Although the time evolution is, in principle, governed by the linear QSD equation (3), it is much more efficient to use the nonlinear version in numerical simulations since it preserves the norm of the state vector. The nonlinear QSD equation can be written as [18],

$$\begin{aligned}
\frac{\partial}{\partial t} \tilde{\psi}_t &= -iH_{\text{sys}} \tilde{\psi}_t + (L - \langle L \rangle_t) \tilde{z}_t^* \tilde{\psi}_t \\
&\quad - [(L^\dagger - \langle L^\dagger \rangle_t) \bar{O} - \langle (L^\dagger - \langle L^\dagger \rangle_t) \bar{O} \rangle] \tilde{\psi}_t,
\end{aligned} \tag{14}$$

where  $\tilde{\psi}_t = \frac{\psi_t}{\|\psi_t\|}$  is the normalized state vector,  $\langle L \rangle_t = \langle \tilde{\psi}_t | L | \tilde{\psi}_t \rangle$  denotes the mean value of the operator and  $\tilde{z}_t^* = z_t^* + \int_0^t ds \alpha^*(t, s) \langle L^\dagger \rangle_s$  is the shifted noise.

Let us emphasize that our results are valid for an arbitrary correlation functions. For simplicity, in this paper, we model the noise as an Ornstein-Uhlenbeck process

for which the correlation function is  $\alpha(t, s) = \frac{\gamma}{2} e^{-\gamma|t-s|}$ . This correlation function is very useful in demonstrating the non-Markovian effects as well as the Markov case by varying the parameter  $\gamma$ , which controls the memory effect of the reservoir. When  $\gamma \rightarrow \infty$ ,  $\alpha(t, s) \rightarrow \delta(t - s)$  it corresponds to the Markov case. Oppositely, when  $\gamma$  is small, the system is in a non-Markovian regime.

### III. NUMERICAL RESULTS AND DISCUSSIONS

Having developed the time-local non-Markovian QSD equation for the two interacting qubits, this section is concerned with the non-Markovian dynamics of entanglement involving entanglement generation and evolution [31–39]. Throughout the paper, the degree of entanglement is measured by "concurrence" [40].

#### A. Memory-assisted entanglement generation

With our exact non-Markovian QSD equation (14), we are able to consider the important non-Markovian feature of entanglement generation. For this purpose, we consider the separable state  $\psi_0 = |11\rangle$  to show how the environmental memory affects the entanglement generation. It has been known that the entanglement may be generated for a qubit system through coupling qubits to a common bath. However, it is not clear how the environmental correlation times affects entanglement generation. In Fig. 1, we plot the entanglement evolution for different parameters of  $\gamma$ , where  $1/\gamma$  gives the memory time of the environment. With this initial state, we can see that when  $\gamma \geq 2$  (close to the Markov limit for this model), there will be no visible entanglement generated for all times. In fact, it is easy to show that in the Markov limit no entanglement will be generated (i.e.  $\gamma \rightarrow \infty$ ) for this special initial state. In addition, it is seen that when the parameter is too small, the entanglement generation becomes less significant shown in our numerical

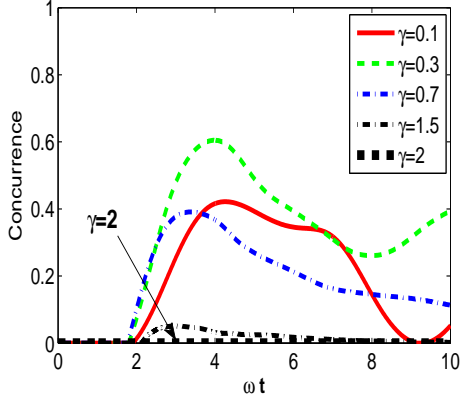


FIG. 1: Entanglement generation for different memory times of the noise. A particular initial state is chosen as  $\psi_0 = |11\rangle$ . The curves for different  $\gamma$  are marked in the graph. The other parameters are  $\omega_A = \omega_B = 0.5\omega$ ,  $\kappa_A = \kappa_B = 1$ ,  $J_{xy} = 0.5$ .

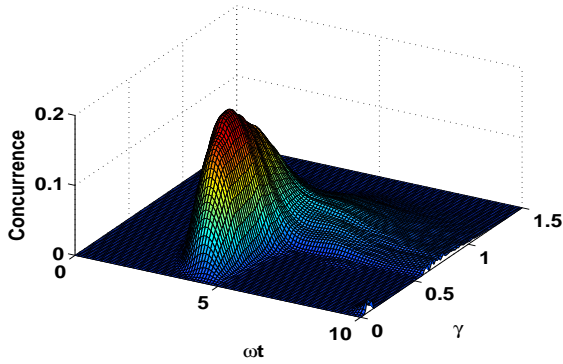


FIG. 2: Entanglement generation vs  $\gamma$  and  $t$ . A particular initial state is chosen as  $\psi_0 = |11\rangle$ . The other parameters are  $\omega_A = \omega_B = 0.5\omega$ ,  $\kappa_1 = \kappa_B = 1$ ,  $J_{xy} = J_z = 0$ .

simulations. This shows that the non-Markovian property affects generating entanglement in a subtle way.

We can show that entanglement generation for this special initial state is due to the environmental memory rather than the spin-spin coupling. Interestingly enough, we find that the largest amount of entanglement may be generated if the correlation time of environment is neither too long ( $\gamma$  is small) nor too short ( $\gamma$  is large). In Fig. 2, we plot the case without coupling between the two spins. The results confirm that entanglement for the two non-interacting qubits is purely generated by environment memory. Our results here have clearly shown that non-Markovian noise can be very useful rather than harmful to the entanglement if the memory times are properly chosen.

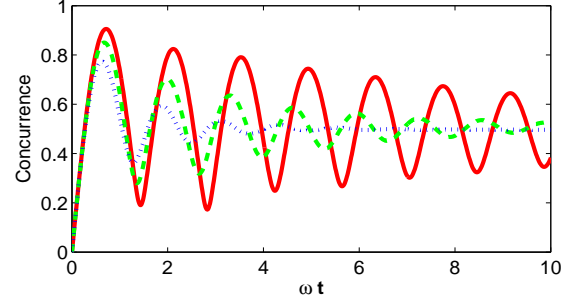


FIG. 3: Entanglement generation. The red (solid), green (dashed) and blue (short-dashed) lines represents  $\gamma = 0.5$ ,  $\gamma = 1$ ,  $\gamma = 2$ , respectively. The initial state is  $\psi_0 = |10\rangle$ . The other parameters are  $\omega_A = \omega_B = 0.5\omega$ ,  $J_{xy} = 0.5$ ,  $J_z = 0.1$ ,  $\kappa_A = \kappa_B = 1$ .

## B. Steady state entanglement

Quantum trajectories generated by the non-Markovian QSD equation (3) have many interesting properties. For instance, we can use those quantum trajectories to study the steady state entanglement. In Fig. 3, we plot the entanglement dynamics from a separable initial state  $\psi_0 = |10\rangle$ . The numerical results show that the initial pure state evolves into a steady entangled state. In fact, with the help of non-Markovian trajectories, we can show that all the trajectories will localize to the eigenstates of the Lindblad operator  $L$  and the residual entanglement of the steady state is not zero. Now, let us examine the large  $t$  behavior of the single trajectories. First, we choose the following four states

$$\begin{aligned} |\psi_1\rangle &= |11\rangle, \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle), \\ |\psi_4\rangle &= |00\rangle. \end{aligned} \quad (15)$$

which form a complete basis of the Hilbert space. Hence, an arbitrary state can always be written as

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle + c_3 |\psi_3\rangle + c_4 |\psi_4\rangle, \quad (16)$$

where  $|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 = 1$ .

Substituting Eq. (16) into the QSD Eq. (12), we get the following stochastic differential equations for the co-

efficients:

$$\frac{\partial}{\partial t}c_1(t) = i(2\omega + J_z)c_1 - 2F_1(t)c_1 - 2F_3(t)c_1, \quad (17)$$

$$\frac{\partial}{\partial t}c_2(t) = i(J_{xy} - J_z)c_2 + \sqrt{2}z_t^*c_1 - 2F_1(t)c_2 + 2F_3(t)c_2 - 2\sqrt{2}\tilde{F}_5(t, z^*)c_1, \quad (18)$$

$$\frac{\partial}{\partial t}c_3(t) = -i(J_{xy} + J_z)c_3, \quad (19)$$

$$\frac{\partial}{\partial t}c_4(t) = i(J_z - 2\omega)c_4 - \sqrt{2}z_t^*c_2. \quad (20)$$

where  $\tilde{F}_5(t, z^*) = i \int_0^t ds' F_5(t, s') z_{s'}^*$ . Here, we assume that the two qubits are identical and symmetric ( $\omega_A = \omega_B, \kappa_A = \kappa_B$ ) for simplicity. The more general case can be dealt with in a similar way. For the initial state  $\psi_0 = |10\rangle$ , the initial conditions for this set of equations are

$$c_1(0) = 0, \quad (21)$$

$$c_2(0) = \frac{1}{\sqrt{2}}, \quad (22)$$

$$c_3(0) = \frac{1}{\sqrt{2}}, \quad (23)$$

$$c_4(0) = 0. \quad (24)$$

For the above initial conditions,  $c_1(t) \equiv 0$  at any times, and the steady state solution for  $c_2(t)$  is simply given by

$$c_2(\infty) = 0. \quad (25)$$

Thus, the large  $t$  state for a single trajectory will not contain  $|\psi_1\rangle$  and  $|\psi_2\rangle$  components, i.e.

$$|\psi_{\text{final}}\rangle = c_3 |\psi_3\rangle + c_4 |\psi_4\rangle. \quad (26)$$

Now, it is clear that all trajectories will always evolve into a sub-space of the Hilbert space spanned by  $|\psi_3\rangle$  and  $|\psi_4\rangle$ , which are the eigenstates of the Lindblad operator  $L = \sigma_-^A + \sigma_-^B$ . Moreover, we can easily derive the equations of the motion for the ensemble means of the cross terms. For example,  $M[c_3^*c_4]$  satisfies the following equation:

$$\frac{d}{dt}M[c_3^*c_4] = (iJ_{xy} + 2iJ_z - 2i\omega)M[c_3^*c_4] - \sqrt{2}M[z_t^*c_3^*c_2].$$

Since  $c_2(\infty) = 0$  in the long-time limit, we have

$$M[c_3^*c_4] \rightarrow 0, \quad M[c_3^*c_4] \rightarrow 0. \quad (27)$$

where  $t$  is large. Therefore, the density matrix recovered by  $\rho_t = M[|\psi_t(z^*)\rangle\langle\psi_t(z^*)|]$  will approach to the steady state density matrix in long-time limit:

$$\rho_s \approx \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & M[c_3^*c_3]|_{t \rightarrow \infty} & 0 \\ 0 & 0 & 0 & M[c_4^*c_4]|_{t \rightarrow \infty} \end{bmatrix}, \quad (28)$$

in the basis  $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle\}$ . The steady state density matrix in the basis  $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$  is then

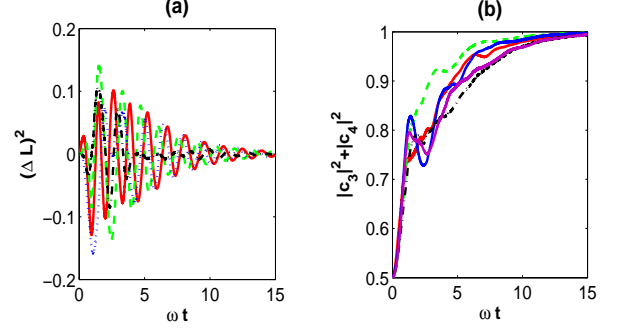


FIG. 4: Time evolution of the fluctuation  $(\Delta L)^2$  and the probability  $|c_3|^2 + |c_4|^2$  for quantum trajectories. Each curve represents a single quantum trajectory. The parameters are  $\omega_A = \omega_B = 0.5\omega$ ,  $\kappa_A = \kappa_B = 1$ ,  $\gamma = 1$ ,  $J_{xy} = 0.5$ .

given by

$$\rho_s \approx \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{bmatrix}. \quad (29)$$

where  $\rho_{22} = \rho_{33} = \frac{1}{2}M[c_3^*c_3]|_{t \rightarrow \infty}$ ,  $\rho_{23} = -\frac{1}{2}M[c_3^*c_4]|_{t \rightarrow \infty}$  and  $\rho_{44} = M[c_4^*c_4]|_{t \rightarrow \infty}$ .

The analytical results shown above can be verified by our numerical simulation as illustrated in the figure 4.

In Fig. 4 (a), we plot the fluctuation of the system Lindblad operator  $L$  defined as  $(\Delta L)^2 = \langle L^2 \rangle - \langle L \rangle^2$ . It is shown that, as time evolves, the fluctuation of  $L$  converge to zero in the final steady state for all trajectories. In Fig. 4 (b), we plot the sum of the probabilities of  $|\psi_3\rangle$  and  $|\psi_4\rangle$  for a few trajectories. The sum of the probability  $|c_3|^2 + |c_4|^2$  will always evolve to 1 for any single trajectories which means  $c_1$  and  $c_2$  are always zero in the final steady state. The numerical results in Fig. 4 simply show that the final state of single trajectories take the form of Eq. (26) as predicted by analytical analysis. In addition, our numerical results show that the final density matrix is approximately given by

$$\rho_{\text{final}} \approx \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.25 & -0.25 & 0 \\ 0 & -0.25 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}. \quad (30)$$

which is also consistent with our prediction in Eq. (29).

It should be noted that such a steady state has been shown to be the stationary solution of the Markov master equation [41, 42]. In our QSD method, it is showed that this state is also the stationary solution of the non-Markovian quantum trajectories. Therefore, the ensemble mean of the quantum trajectories recovers the non-Markovian steady state of the density matrix. It is important to note that our non-Markovian theory does not

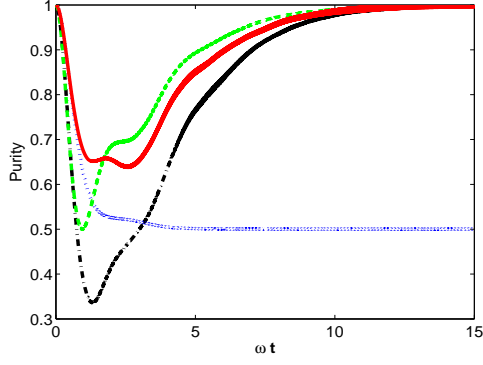


FIG. 5: Time evolution of purity for different initial states. The initial states for red (solid), green (dashed), blue (short-dashed) and black (dash-dotted) lines are  $\psi_0 = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$ ,  $\psi_0 = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ ,  $\psi_0 = |10\rangle$ , and  $\psi_0 = |11\rangle$ , respectively. The other parameters are  $\omega_A = \omega_B = 0.5\omega$ ,  $\kappa_A = \kappa_B = 1$ ,  $\gamma = 1$ ,  $J_{xy} = 0.7$ ,  $J_z = 0.3$ .

depend on the existence of the non-Markovian master equation.

Finally, we present the time evolution of the purity, defined as  $Purity = tr(\rho^2)$ , which determines whether the quantum state is pure or mixed. Specifically, we find the purity of the initial state  $\psi_0 = |10\rangle$ , which evolves into a steady state with some residue entanglement preserved in the final steady state. In Fig. 5, it is validated that the final steady state is a mixed entangled state since the purity is smaller than one at the steady state. The purity is approximately 0.5 in long time limit, which is consistent with Eq. (30). For all the other three initial states, we will see two qubits will firstly evolve into a mixed (perhaps entangled) state by the coupling interaction, then decay into the final ground state  $|00\rangle$  (which is a pure state) due to the dissipative environment.

### C. Exact versus approximate QSD equations

For the coupled two-qubit model, the exact  $O$  operator contains five terms, as shown in Eq. (8). It is interesting to investigate the perturbative QSD equations based on the exact expression of the  $O$  operator. For example, we may drop the first order noise-dependent term  $O_5$  and only keep the first four terms, i.e. the  $O$  operator in Eq. (8) is approximately written as

$$O(t, s, z^*) \approx f_1(t, s)O_1 + f_2(t, s)O_2 + f_3(t, s)O_3 + f_4(t, s)O_4. \quad (31)$$

Clearly, the first order term  $O_5$  becomes less important when the system is close to the Markov regimes. Also, numerical calculations will be greatly simplified if the first order noise is neglected. In Fig. 6, we have compared the exact time evolution of entanglement and the approximate case without  $O_5$ . The numerical result shows that

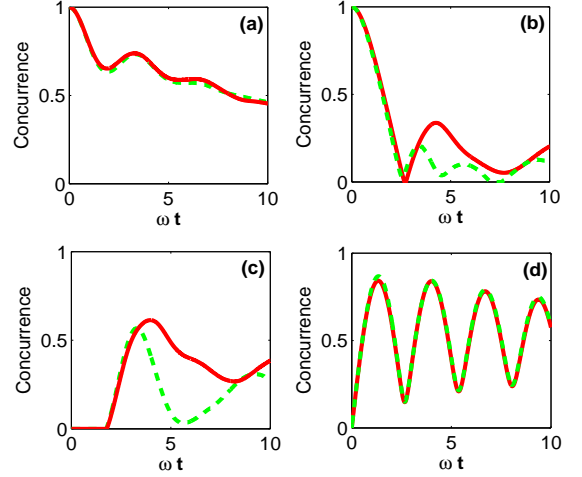


FIG. 6: Comparison of the dynamics generated by the approximate  $O$  operator and the exact  $O$  operator. The red (solid) curve represents the exact QSD equation, and the green (dashed) curve is for the approximate QSD. (a) (b) (c) and (d) represent four different initial states  $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ ,  $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$ ,  $|11\rangle$  and  $|10\rangle$ , respectively. The parameters are chosen as  $\gamma = 0.3$ ,  $\omega_A = \omega_B = 0.5\omega$ ,  $\kappa_1 = \kappa_B = 1$ ,  $J_{xy} = 0.5$ .

the accuracy of the zeroth order approximation is sensitively dependent on the initial state of the qubits. For the anti-correlated initial states, Fig. 6 (a) and (d) shows that the approximation is extremely good compared with the exact result. On the other hand, for the correlated initial states in case (b) and (c), the noise dependent term tends to smooth out the entanglement curve. In fact, this can be explained by analyzing the QSD Eq. (17-20). If the initial states do not contain  $\psi_1$  component, i.e.  $c_1(0) = \langle \psi_0 | \psi_1 \rangle = 0$ , then  $c_1(t)$  is always zero during the time evolution. The fifth term of the  $O$  operator  $O_5 = 2\sigma_-^A \sigma_-^B$  will not affect the dynamics of the system, since the coefficient of  $O_5$ , i.e.  $\tilde{F}_5(t, z^*)$ , is always associated with  $c_1(t)$ . This observation explains our results plotted in Fig. 6. As shown in Fig. 6, the noiseless approximation (setting  $O_5 = 0$ ) gives extremely good results for the two initial states which do not contain  $|11\rangle$  component. Our simple analysis here has already shown the complexity level of the approximate QSD equations arising from the noise expansion of the  $O$  operator. More in-depth study on the approximate QSD equations will be presented elsewhere.

## IV. CONCLUSION

In conclusion, the central aim of this paper is to develop a time-local theory of non-Markovian quantum state diffusion for a model consisting of two interacting qubits – a model that is of interest for both quantum information processing and quantum optics. As an application, we used the non-Markovian quantum trajectory to analyze the entanglement evolution of the qubit system for several parameters ranging from strongly non-Markovian to Markov regimes. Our results also showed explicitly how the environmental memory generates the entanglement of the qubit system. With the non-Markovian QSD equation, we further studied the entanglement evolution and the steady entangled state for

the QSD equation. Our time-local QSD approach may be extended to multiple qubit systems, this will be the topic for a future publication.

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